where cosecϕ is the ratio of d to the difference in the moon and spacecraft radii (see Fig. 4).

From Eqs. (6, 7 and 10) it can be shown that contours of constant V_f in a (d, ϕ) coordinate system are circles, with center $(\mu \cot\gamma/(V_f^2 - V_M^2), 0)$ and radius

$$\mu[\{(V_f^2 - V_m^2)\tan\gamma\}^{-2} + \{2(V_f^2 - V_m^2)V_m^2 \sin^2\gamma\}^{-1} + \{2V_m \sin\gamma\}^{-4}]^{1/2}$$

Examples of these circles for $\gamma = 20°$, $40°$ and $90°$ are shown in scale with the moon in Fig. 4. The "dashed" circular arcs intersecting the constant V_f circles, are the boundary of the moon impact region. Thus, the regions with V_f greater than a fixed value are crescent-shaped. The shading on the moon indicates its velocity with respect to the spacecaft. The moon's direction is indicated by it being shaded as if it were traveling away from the sun; and the fraction of disc shaded gives the speed as a fraction of $2V_M$.

For the high velocity producing orbits, the spacecraft has to be directed towards a small region just behind the moon with $\gamma \sim 40°$. For $\gamma \sim 20°$ the escape region (i.e., $V_f \geq V_E$) becomes large; in Fig. 4 it is shown to be a region of over 10,000 km in diameter.

The "head on" conditions when $\gamma = 90°$, permits good direction control, but the extent of the escape region is small.

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Stability of a Model Reference Control System

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Introduction

In a recent paper Lindh and Likins[1] compared the so-called infinite determinant method and a numerical implementation of Floquet theory for obtaining the regions of the parameter space corresponding to stability and instability of the null solution of a restricted class of linear, periodic coefficients, ordinary homogeneous differential equations. In this Note these methods will be applied to examine the stability of a model reference adaptive control system having sinusoidal input.

In recent years model reference adaptive control systems have proved very popular, particularly for practical applications to devices such as auto-pilots where rapid adaptation is required. The basic idea is shown in Fig. 1. The input $\theta_i(t)$ to the system is also fed to a reference model, the output of which is proportional to the desired response; the outputs of the model and system are then differenced to form an error

$$e(t) = \theta_m(t) - \theta_s(t) \tag{1}$$

Since the error is to be zero when the system is in the optimum state it is used as a demand signal for the adaptive loops which adjusts the variable parameters in the system to the desired value.

Various methods of synthesizing the adaptive loops have been proposed but the one that has proved most popular was that developed by Whittaker et al.[2] at the Massachusetts Institute of Technology and referred to as the "M.I.T." rule. Here the performance criterion is taken as the integral of error squared and a heuristic argument is given for reducing this over an unspecified period of time. This leads to a rule that a particular parameter K_i should be adjusted so that

$$\dot{K}_i = -Ge(\partial e/\partial K_i) \tag{2}$$

where G is the constant gain.

Although the "M.I.T." rule results in practically realizable systems, mathematical analysis of the adaptive loops, even for simple inputs, prove to be very difficult and it is usual in the design process to carry out much analogue computer simulation. The system equations are nonlinear and nonautonomous and since the nonlinearity is of the multiplicative kind, the mass of theory on instantaneous nonlinearities associated with the names of Lur̂e and Popov, in particular, is not applicable. In order to point out some of the difficulties we shall consider a simple first-order system having sinusoidal input.

Adaptive Control System

Since the intention, as previously mentioned, is to point out the difficulties involved in a stability investigation of a model reference adaptive control system, a simple first-order system with controllable gain will be considered.

Consider a model and system to be governed respectively by the equations

$$T\dot{\theta}_m(t) + \theta_m(t) = K\theta_i(t) \tag{3a}$$

$$T\dot{\theta}_s(t) + \theta_s(t) = K_vK_c\theta_i(t) \tag{3b}$$

where a dot denotes differentiation with respect to time t; the time constant T and model gain K are constant and known, but the process gain K_v is unknown and possibly time varying. The problem here is to determine a suitable adaptive loop to control K_c so that K_vK_c eventually equals the model gain K. The "M.I.T." rule gives

$$\dot{K}_c = -Ge(\partial e/\partial K_c) = Be\theta_m \tag{4}$$

where $B = GK_v/K$, and this leads to the scheme of Fig. 2.

If a sinusoidal input of magnitude $R \sin\omega t$ is applied at $t = 0$, when $\theta_m(t)$, $\theta_s(t)$ are zero and $K_vK_e \neq K$ and if subsequently K_v remains constant but K_c is adjusted according to Eq. (4), then using Eqs. (1, 3 and 4) the system equations become

$$T\dot{e}(t) + e(t) = (K - K_vK_c)R \sin\omega t \tag{5a}$$

$$\dot{K}_c = Be(t)\theta_m(t) \tag{5b}$$

Now consider that the adaption is switched on when the model response $\theta_m(t)$ has reached its steady-state value $\theta_{ms}(t)$ given by

$$\theta_{ms}(t) = [KR/(1 + T^2\omega^2)](\sin\omega t - T\omega \cos\omega t)$$

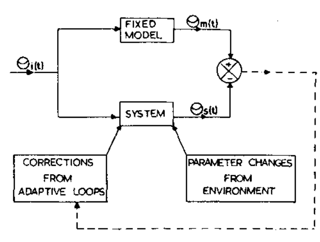

Fig. 1 Model reference adaptive control system.

Received July 21, 1970; revision received December 22, 1970. The author would like to express his gratitude to P. C. Parks of the Institute of Engineering Control, University of Warwick for his useful discussions concerning this problem. Thanks are due also to D. G. Evans and K. Underdown for carrying out the analogue computer simulation.

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then Eqs. (5) become

$$T\dot{e}(t) + e(t) = x(t)\sin\omega t \quad (6a)$$

$$\dot{x}(t) = -[K_vBKR/(1 + T^2\omega^2)](\sin\omega t - \omega T\cos\omega t)e(t) \quad (6b)$$

where $x(t) = K - K_vK_c$.

Introducing the dimensionless parameters $\Pi_1 = \omega T$, $\Pi_2 = TR^2KK_vB$ and the dimensionless variables $\tau = \omega t$, $\xi_1 = e/KR$, $\xi_2 = -x/K$, Eqs. (6) may be written in the nondimensional form

$$\frac{d}{d\tau}\begin{bmatrix}\xi_1(\tau)\\ \xi_2(\tau)\end{bmatrix} = \begin{bmatrix} -\dfrac{1}{\Pi_1} & -\dfrac{1}{\Pi_1}\sin\tau \\ \dfrac{\Pi_2}{\Pi_1(1+\Pi_1^2)}(\sin\tau - \Pi_1\cos\tau) & 0\end{bmatrix}\begin{bmatrix}\xi_1(\tau)\\ \xi_2(\tau)\end{bmatrix} \quad (7)$$

which is a linear matrix differential equation of the form $\boldsymbol{\xi}' = \mathbf{A}(\tau)\boldsymbol{\xi}$ with $\mathbf{A}(\tau)$ periodic in τ with period 2Π (a prime denotes differentiation with respect to τ).

Stability Theory

We are interested in finding the domains of the parameter space for which the null solution of the system of first-order equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \quad (8)$$

$$\mathbf{A}(t + T) = \mathbf{A}(t) \quad (9)$$

is stable, where $\mathbf{x}(t)$ is an n vector and $\mathbf{A}(t)$ an $n \times n$ matrix of period T in t. This problem was discussed in some detail in Ref. 1 and we shall confine ourselves here to a brief summary.

One approach is a numerical implementation of Floquet theory. From Floquet theory it can be shown that for system (8), subject to condition (9), there exists a constant $n \times n$ matrix $\mathbf{C}$, known as the monodromy matrix of the system, such that

$$\mathbf{x}(t + T) = \mathbf{C}\mathbf{x}(t) \quad (10)$$

Using a Liapunov type transformation it then follows that a necessary and sufficient condition for the null solution of the system to be uniformly asymptotically stable is that all the eigenvalues of the monodromy matrix $\mathbf{C}$ lie within the unit circle $|z| < 1$. If the eigenvalues of the monodromy matrix lie in the circle $|z| \leq 1$ and the eigenvalues on $|z| = 1$ correspond to unidimensional Jordan cells, then the null solution is uniformly stable.[3]

In practice the monodromy matrix $\mathbf{C}$ is obtained by integrating Eqs. (8) numerically over a period.[4,5] [A check on the value of $\mathbf{C}$ may be employed since $\det\mathbf{C} = \exp\int_0^T \mathrm{trace}\mathbf{A}(t)dt$.] This is followed by a numerical evaluation of its eigenvalues thus giving an assessment of stability or instability. However, if one is only interested in the question of stability, the last step may be dispensed with. Instead the characteristic polynomial of $\mathbf{C}$ may be obtained using the Faddeeva[6] algorithm followed by a stability assessment using the determinant method of Jury.[7] This procedure has been used satisfactorily by the author[4] and since it involves only matrix multiplication and the evaluation of second-order determinants it gives a considerable saving in computational time over direct evaluation of the eigenvalues.

An alternative method of obtaining the transition boundaries, between stable and unstable regions in parameter space, is the so-called infinite determinant method. This method is restrictive in its use since it requires the form of the solutions on the transition boundaries to be known. Assuming the continuous dependence of stability on parameter values it follows that on any transition boundary there exists an eigenvalue λ_i of the monodromy matrix such that its modulus is

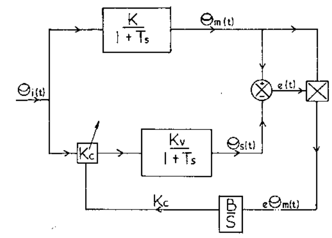

Fig. 2 First-order system—M.I.T. gain adaptation.

unity. Thus on the transition boundaries there must exist[8] an almost periodic solution of the form

$$\mathbf{x}_i(t) = e^{j(\arg\lambda_i)t/T}\mathbf{p}_i(t) \quad (11)$$

where $\mathbf{p}_i(t)$ is a periodic n vector with period T (note that in general such solutions may exist within stable or unstable regions but not within regions of uniform asymptotic stability). If the monodromy matrix of the system is symplectic then its characteristic equation is reciprocal, and the form of the solutions on the transition boundaries for such systems is discussed in detail in Ref. (1).

For certain systems (e.g., uncoupled canonical systems) it can be shown that the transition boundaries are characterized by the existence of solutions of Period T or a restricted class of functions of period $2T$. The procedure then is to assume Fourier series developments with undetermined coefficients, for these solutions; these solutions are then substituted into the system equations and the principle of harmonic balance employed to obtain an infinite system of simultaneous, linear, homogeneous algebraic equations for the coefficients. For those values of the parameters which admit the assumed periodic solutions the homogeneous algebraic equations must have a nontrivial solution and this is the case only if the infinite determinant (Hill determinant) of the coefficients is zero. In practice the Fourier series is truncated and the corresponding Hill determinant solved to give lines in parameter space. If the truncation point of the Fourier series is extended and the zeros of the corresponding Hill determinants of increasing order converge to some limit set of lines then the infinite determinant procedure is said to be convergent, the convergent set of lines in parameter space being the required transition boundary between stable and unstable regions.

Application of Theory to Adaptive Control System

Applying the numerical implementation of Floquet theory to system (7) stability boundaries in the parameter space $\Pi_1 - \Pi_2$ were obtained. These stability boundaries are shown in Fig. 3 and in the main they have been verified by analogue computer simulation.

Although system (7) is not canonical it can be shown that the only solution corresponding to transition boundaries

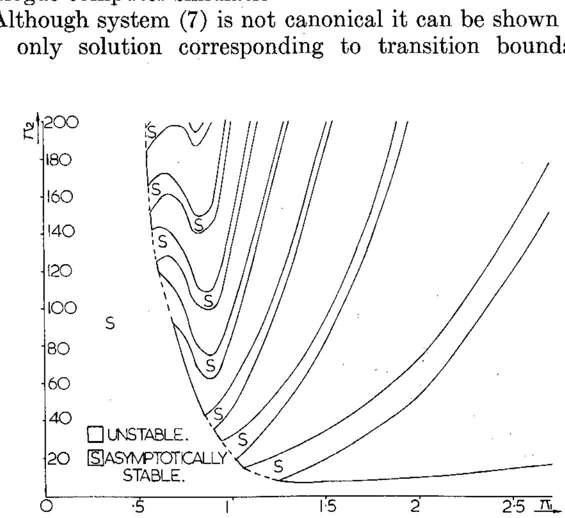

Fig. 3 Stability regions in parameter space.

from stable to unstable regions are those of period T and $2T$ (2π and 4π in this case). The monodromy matrix $\mathbf{C}$ of Eq. (7) is such that

$$\det\mathbf{C} = \exp\left\{\int_0^{2\pi} \text{trace}\mathbf{A}(\tau)d\tau\right\} = \exp\{-2\pi/\Pi_1\}$$

so that if λ_1, λ_2 are the eigenvalues of C then

$$\lambda_1\lambda_2 = \exp\{-2\pi/\Pi_1\} = \rho^2 \text{ (say)} \qquad (12)$$

If λ_1, λ_2 are complex conjugates then it follows from Eq. (12) that they lie on a circle of radius ρ so that complex roots cannot have modulii unity since this would imply $\rho = 1$ which is only possible if Π_1 is infinite. Thus on the transition boundary there must exist a real root having the value of $+1$ or -1 (note that real roots λ_1, λ_2 are inverse points with respect to the circle of radius ρ). It follows from Eq. (11) that the transition boundary is characterized by a solution of period T or $2T$.

On substituting a Fourier series, with undetermined coefficients, of period 4π in Eqs. (7) and balancing like terms, it can be shown, by induction, that the corresponding Hill determinants are sums of squares and therefore cannot be zero for any values of the parameters. This has been verified by analogue computer simulation and by the results of the Floquet theory analysis.

In the case of the harmonic solution, of period 2π, substituting the Fourier series

$$\xi_1 = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\tau + b_n \sin n\tau)$$

into Eqs. (7) [the corresponding series for ξ_2 being obtained by the second equation of (7)] and balancing the terms leads to two distinct sets of linear homogeneous algebraic equations for the coefficients (a_{2n},b_{2n}), (a_{2n+1},b_{2n+1}), ($n = 0,1,2$, etc.), respectively. The corresponding Hill determinants of order r, in each case, are polynomials of order r in Π_2 having coefficients which are functions of Π_1. For a particular r these polynomials are solved for a range of values of Π_2 and the zeros plotted to obtain the transition boundary in the parameter space. The value of r is then increased and the corresponding Hill determinants solved until a convergent set of boundaries are obtained. It is found that for $\Pi_1 > 1.5$, where the enveloping boundary is continuous, consideration of fifth order Hill determinants is sufficient but for $\Pi_1 < 1.5$ where the enveloping boundary is discontinuous, the method is not found to be very satisfactory. Hill determinants of order eleven have to be considered before a true picture begins to emerge and the order has to be increased still further before an enveloping boundary is obtained to a satisfactory degree of accuracy. For this problem the region $\Pi_1 < 1.5$ is important since it is the most likely range of application in practice.

Conclusions

In this Note the stability regions in nondimensional space have been obtained for a first-order controllable gain model reference adaptive control system, and the results illustrate the complexity of the question of stability when dealing with such systems.

Both a numerical implementation of Floquet analysis and the infinite determinant method of analyzing linear differential equations with periodic coefficients have been employed. The infinite determinant approach was not found to be very satisfactory in the region of parameter space where the stability boundaries are complex in nature. Since, when dealing with linear differential equations with periodic coefficients, complex stability boundaries frequently occur, it throws some doubt on the performance of the method in general. Although the Floquet analysis involved investigating the eigenvalues of the monodromy matrix at a network of points in parameter space the results obtained were far more satisfactory, and for the particular problem considered the computation time was less. For higher order systems the use of the Faddeev algorithm and the Jury procedure would further reduce computational time when using the Floquet approach.

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Some Considerations of a Simplified Velocity Spectrum Relation for Isotropic Turbulence

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THE "three-dimensional" velocity spectrum function, $E(k)$, defined such that

$$\frac{3}{2}\hat{u}^2 = \int_0^\infty E(k)dk \qquad (1)$$

($\hat{u}$ = rms velocity fluctuation level, k = wave number), is of interest in both theoretical and practical studies of turbulence phenomena. A simple form for $E(k)$ was proposed by von Kármán[1]

$$k_eE(k)/\hat{u}^2 \propto (k/k_e)^4[1 + (k/k_e)^2]^{-17/6} \qquad (2)$$

(k_e = energy-containing wave number, as an "interpolation" formula joining the range $k \approx 0(E(k) \propto k^4)$ to the inertial subrange, wherein $E(k) \propto k^{-5/3}$ for $k \gg k_e$ in Eq. (2).

At high-wave numbers, of the order of the Kolmogoroff wave number $k_K \doteq L_K^{-1}$, where

$$L_K = (\nu^3/\Phi)^{1/4} \qquad (3)$$

the spectrum function is "cut-off" by viscous effects. The inadequacy of Eq. (2) for this wave number range is reflected

Received July 27, 1970; revision received February 9, 1971. This research was supported by the Advanced Research Projects Agency of the Department of Defense and Space and Missiles Systems Organization, Air Force Systems Command and was monitored by Space and Missiles Systems Organization, Air Force Systems Command, under Contract F04-701-70-C-0128.

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